Matrix Equations and Applications

Finite Math

8 November 2018

Quiz

Which is better?

Matrix Equations

Theorem

Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then

- *Addition Properties*
  1. *Associative*
  $$(A+B)+C = A+(B+C)$$
  2. *Commutative*
  $$A+B = B+A$$
  3. *Additive Identity*
  $$A+0 = 0+A = A$$
  4. *Additive Inverse*
  $$A+(-A) = (-A)+A = 0$$

Matrix Equations

Theorem

Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then

- *Multiplication Properties*
  1. *Associative Property*
  $$A(BC) = (AB)C$$
  2. *Multiplicative Identity*
  $$AI = IA = A$$
  3. *Multiplicative Inverse*
  If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

Matrix Equations

Theorem

Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then

- *Combined Properties*
  1. *Left Distributive*
  $$A(B+C) = AB + AC$$
  2. *Right Distributive*
  $$(B+C)A = BA + CA$$

Matrix Equations

Theorem

Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then

- *Equality*
  1. *Addition*
     If $A = B$, then $A + C = B + C$
  2. *Left Multiplication*
     If $A = B$, then $CA = CB$
  3. *Right Multiplication*
     If $A = B$, then $AC = BC$

Solving Matrix Equations

We can use the rules above to solve various matrix equations. In the next 3 examples, we will assume all necessary inverses exists.

Example

Suppose A is an $n \times n$ matrix and B and X are $n \times 1$ column matrices. Solve the matrix equation for X

$$AX = B.$$

Example

Suppose A is an $n \times n$ matrix and B, C, and X are $n \times 1$ matrices. Solve the matrix equation for X

$$AX + C = B.$$

Now You Try It!

Example

Suppose A and B are $n \times n$ matrices and C is an $n \times 1$ matrix. Solve the matrix equation for X

$$AX - BX = C.$$

What size matrix is X?

Matrix Equations and Systems of Linear Equations

We can also solve systems of equations using the above ideas. These apply in the case that the system has the same number of variables as equations and the coefficient matrix of the system is invertible. If that is the case, for the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & a_{1n}x_n & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

we can create the matrix equation

$$AX = B$$

Matrix Equations and Systems of Linear Equations

we can create the matrix equation

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{1n} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, if A is invertible (as is the case when the system is consistent and independent, i.e., exactly one solution), we have

$$X = A^{-1}B.$$

Solving Systems of Equations Using Matrices

Example

Solve the system of equations using matrix methods

$$\begin{array}{rcrcl} x & + & 2y & = & k_1 \\ x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 1$, $k_2 = 3$

(b) $k_1 = 3$, $k_2 = 5$

(c) $k_1 = -2$, $k_2 = 1$

Now You Try It!

Example

Solve the system of equations using matrix methods

$$\begin{array}{rcrcl} 2x & + & y & = & k_1 \\ 5x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 2, k_2 = 13$

(b) $k_1 = 2, k_2 = 4$

(c) $k_1 = 1, k_2 = -3$

Solution

(a) $x = -7$ *and* $y = 16$, *(b)* $x = 2$ *and* $y = -2$, *(c)* $x = 6$ *and* $y = -11$

Two-Industry Model

To simplify the ideas, we will assume we are in an economy with only two industries: coal and steel. In this economy, to produce \$1 worth of coal requires an input of \$0.10 from the coal sector and \$0.20 from the steel sector; and to produce \$1 worth of steel requires an input of \$0.20 from the coal sector and \$0.40 from the steel sector. The final demand (the demand from all other users of coal and steel) is \$20 billion for coal and \$10 billion for steel. What we would like to know is how much total coal and steel needs to be produced to meet this final demand.

Two-Industry Model

If the coal and steel sector produces just the final demand of coal and steel, it would require:

Coal

$$0.1(20) + 0.2(10) = \$4 \text{ billion of coal}$$

Steel

$$0.2(20) + 0.4(10) = \$8 \text{ billion of steel}$$

This only leaves \$16 billion of coal and \$2 billion of steel left over to meet that final demand, well below the required amounts.

Two-Industry Model

So, we need to not only meet the final demand, but also the internal demand. To figure out how to do this, we need two variables

$$x = \text{total output from coal industry}$$

$$y = \text{total output from steel industry.}$$

Then the internal demands (amount of coal and steel required to produce x amount of coal and y amount of steel) are as follows:

Coal

$$0.1x + 0.2y \text{ internal demand for coal}$$

Steel

$$0.2x + 0.4y \text{ internal demand for steel}$$

Two-Industry Model

So, we can create equations for the total amount of coal and steel required by adding the internal and final demands to get the system of equations:

$$\begin{array}{rcccl} \text{Total} & & \text{Internal} & & \text{Final} \\ \text{output} & & \text{demand} & & \text{demand} \\ x & = & 0.1x + 0.2y & + & 20 \\ y & = & 0.2x + 0.4y & + & 10 \end{array}$$

which we rewrite in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

or using letters for the matrices

$$X = MX + D.$$

D is called the *final demand matrix*, X is called the *output matrix*, and M is called the *technology matrix*.

Two-Industry Model

The technology matrix should be read as the inputs entering from the left and outputs leaving from above. That is,

$$
\begin{array}{ccc}
 & & \begin{array}{cc} \multicolumn{2}{c}{\text{Output}} \\ C & S \\ \uparrow & \uparrow \end{array} \\
\text{Input} & \begin{array}{c} C \to \\ \\ \\ S \to \end{array} & \begin{bmatrix} \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \\ \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \end{bmatrix} = M
\end{array}
$$

(C stands for the coal industry and S for the steel industry).

Two-Industry Model

Now that the individual pieces are understood, let's finish solving the problem. Let's begin by solving the matrix equation first:

$$\begin{aligned} X &= MX + D \\ X - MX &= D \\ (I - M)X &= D \\ X &= (I - M)^{-1} D \end{aligned}$$

(Note that this solution requires $I - M$ to have an inverse!)

Two-Industry Model

Now we actually work this out with the numbers from this problem

$$\begin{aligned}
I - M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ -0.2 & 0.6 \end{bmatrix} \\
(I - M)^{-1} &= \frac{1}{(0.9)(0.6) - (-0.2)(-0.2)} \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \\
X = (I - M)^{-1} D &= \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 28 \\ 26 \end{bmatrix}
\end{aligned}$$

So to meet the internal and final demands, \$28 billion of coal and \$26 billion of steel must be produced.

Outline of Input-Output Analysis Solution

To summarize, these are the steps to solving an input-output analysis problem:

1. Find the technology matrix M and the final demand matrix D.
2. Find $I - M$.
3. Find $(I - M)^{-1}$.
4. Find $X = (I - M)^{-1}D$.
5. Interpret the answer in words.